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Diffusion and spectral dimension on Eden tree

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Abstract. We calculate the eigenspectrum of random walks on the Eden tree in two and three dimensions. From this, we calculate the spectral dimension d_s and the walk dimension d_w and test the scaling relation $d_s=2d_f/d_w$ ($=2d/d_w$ for an Eden tree). Finite-size induced crossovers are observed, whereby the system crosses over from a short-time regime where this relation is violated (particularly in two dimensions) to a long-time regime where the behaviour appears to be complicated and dependent on dimension, even qualitatively.

1. Introduction

Treelike structures arise in many situations in statistical physics, including clusters formed by irreversible growth processes modelled, for example, by diffusion-limited aggregation [1] (which are treelike on large scales) and dielectric breakdown [2]. In this paper, we consider a particularly simple tree structure called Eden tree [3] which is formed on a lattice by a modification of the usual Eden process [4] (in which a *compact* cluster grows) so that an empty site which is a neighbour to more than one occupied site becomes ineligible for occupation. An Eden tree is a *compact* random structure in contrast to the classical Cayley tree which cannot be contained in any finite-dimensional space. It is also a *tree* without any loops as opposed to other compact random structures, e.g., a Voronoi lattice.

The study of diffusion on treelike structures is interesting in that it may present qualitatively different behaviour from structures with many loops (particularly those that connect large branches). The Eden tree model was studied in detail by Dhar and Ramaswamy [3] using various methods. They calculated the spectral dimension d_s [5] for two- and three-dimensional Eden trees by applying the *node counting theorem* [6] and also the random walk dimension d_w [7] by a direct simulation of random walks as well as through the *backbone* statistics of the tree, and discussed them in terms of a *scaling argument*. A surprising conclusion was drawn from these discussions; they concluded that a typical long walk samples only order-one segments of the backbone (including the *dangling ends* attached to this segment) so that the usual scaling relation [8]

$$d_s=2d_f/d_w \qquad (1)$$

does not hold. (Here $d_f=d$ since the Eden trees are compact.) While the calculation of

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d_s (particularly in two dimensions) seemed accurate, the graphs from which d_w was deduced appeared to show considerable curvature. Thus a more accurate determination of these exponents appeared desirable to test the violation of relation (1) better.

The scaling law (1) is expected to be valid in general because it follows by assuming only: (a) the equivalence of the vibrational and diffusional problems (Alexander and Orbach [8]) and (b) that a random walk starting from a given site has a homogeneous probability of visiting any site within the average distance travelled in a given time. Here we are considering long-time limits and the expression *homogeneous* is used in the sense that ratios of the probabilities of visiting different sites within the specified distance are finite. In fact relation (1) is sometimes used as the *definition* of the spectral dimension d_s. In their work [3], Dhar and Ramaswamy essentially reject assumption (b) thereby rejecting the relation for the Eden tree.

This is surprising because it appears at first that any random walk satisfying the detailed balance condition should in principle satisfy condition (b) above. This should be true independent of the structure and thus for example (1) holds for Euclidean lattices in all dimensions and also for critical percolation clusters [9].

In this paper, we re-examine this rather curious observation of [3], using a method which allows a more accurate numerical evaluation of both d_s and d_w from the same data. This is the method of calculating the eigenspectrum of the so-called hopping probability matrix W by Saad's diagonalization method [10] and making use of the Laplace transform relationship between this spectrum and quantities such as the return-to-origin probability and the velocity autocorrelation function [11, 12]. (Here W has elements W_{ij} which are just the probability of a random walker at site j hopping to site i in the next time step.) This corresponds to averaging over *all* random walks starting from every point of the cluster. The method was used previously [12, 13] for two- and three-dimensional percolation clusters very successfully and we expect it to work just as well for the Eden trees.

Thus we first construct an Eden tree of size S by simulation on the square and simple cubic lattices, create the matrix W for a random walk (we use a *blind ant* model where the random walker has an equal probability of hopping between any pair of neighbouring sites), and calculate the eigenvalues and eigenvectors of W for the largest M (typically a few hundred) eigenvalues to very high precision (typically up to six decimal places) using the Arnoldi–Saad algorithm [10]. The eigenvalue density $n(\lambda)$ is then expected [13] to scale as

$$n(\lambda) \sim |\ln \lambda|^{d_s/2-1}. \qquad (2)$$

One can also construct another interesting function $\pi(\lambda) = n(\lambda)a_\lambda(\lambda-1)^2$ [14] from the eigenvalues and eigenvectors of W, where a_λ are the coefficients entering the mean position autocorrelation $\langle r(t)\cdot r(0)\rangle = \Sigma_\lambda a_\lambda \lambda^t$. The coefficients a_λ are determined when the stationary initial distribution is expanded in terms of the eigenvectors of W. This function is expected [12, 13] to scale as

$$\pi(\lambda) \sim |\ln \lambda|^{1-2/d_w}. \qquad (3)$$

2. Numerical results

Our numerical results for $n(\lambda)$ and $\pi(\lambda)$ are plotted in figure 1 and figure 2, respectively. (The values of π plotted have been scaled up by a factor of the size S of the tree compared with the convention used in [13] in order to separate different S in the figure.) The cluster sizes for which the calculations were made are $S=2500$, 5000, 10 000 for the square lattice for which 200, 400, and 200 independent realizations were averaged (respectively), and $S=2500$, 5000 for the simple cubic lattice for which 200 and 400 independent realizations were averaged (respectively).

From figure 1(a) for the square lattice, there is apparently a sharp cluster-size dependent crossover which divides a region of larger $|\ln\lambda|$ (correspoding to shorter time) where an excellent power-law fit can be made, from the smaller $|\ln\lambda|$ region where another power-law fit may also be possible. We will see later that the locations of these crossovers are consistent with the finite size scaling with the walk dimension. Figure 1(b) for the simple cubic lattice gives a much less clear indication of such a crossover, but nonetheless, the sudden flattening of the data in the regions of smallest $|\ln\lambda|$ is consistent with a crossover interpretation. We observe that the right-hand side regions in these figures should give the asymptotic exponent corresponding to (2) as S increases (because this is the region before the finite size effects appear to set in). However, the apparent power-laws in the left-hand side region are still not understood. Moreover, it is even possible that the qualitative behaviour of these crossovers changes when S is made much larger; since in the present analysis we are limited to relatively small Eden trees, we cannot be sure if there is not another true asymptotic behaviour.

Estimating the slopes of the log–log plots in the right-hand side regions and using (2), we obtain

$$d_s=1.22\pm0.02 \qquad (d=2) \tag{4}$$

and

$$d_s=1.32\pm0.02 \qquad (d=3) \tag{5}$$

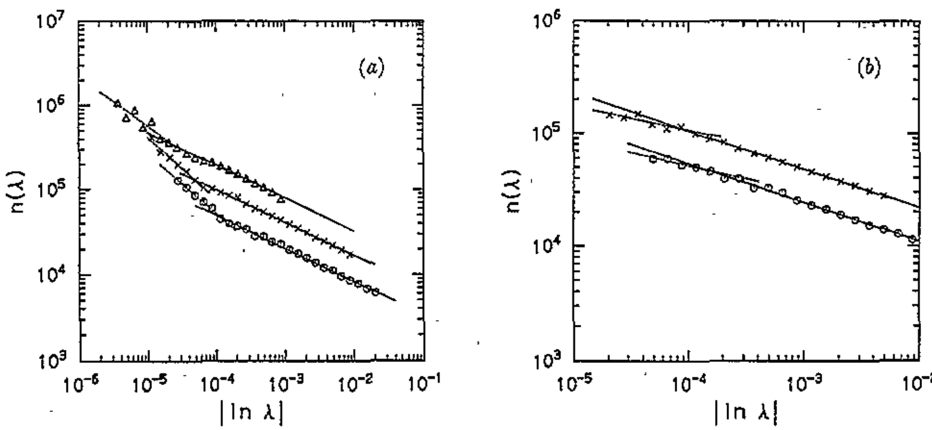

Figure 1. (a) The eigenvalue density $n(\lambda)$ for the Eden tree on the square lattice. The data points ○, ×, and △ correspond to the average over 200, 400, and 200 clusters of size $S=2500$, 5000 and 10 000, respectively. The solid lines are obtained by least squares fitting to the data for $|\ln\lambda|$ larger than its crossover value (number of points used in the fits being 14, 19, and 19 for the three sizes respectively), and the dotted lines are from the similar fits to the data smaller than the crossover point; (b) $n(\lambda)$ for the simple cubic lattice where the symbols ○ and × correspond also to $S=2500$ and 5000, respectively.

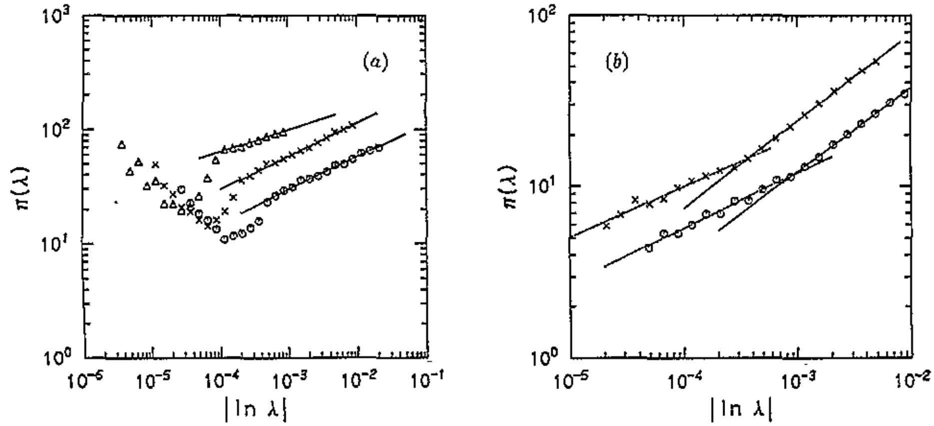

Figure 2. (a) The function $\pi(\lambda)$ for the Eden tree on the square lattice. The symbols ○, ×, and △ have the same meaning as in figure 1; (b) for the simple cubic lattice.

where the errors quoted are mainly the regression errors but also include some error due to the choice of the region to fit and that due to the small variation for different cluster sizes. These estimates are consistent with those of [3] but improves the accuracy especially in three dimensions. In particular, we rule out the possibility that d_s is the same for two and three dimensions.

The results for $\pi(\lambda)$ in figure 2 show an even more interesting cluster-size dependent anomaly. In figure 2(a) for the square lattice, starting from the larger $|\ln \lambda|$ (i.e. shorter time) region, there is a range of about one to two decades where an excellent power law fit can be made, and then a sudden and large decrease occurs. This anomaly is reproducible in independent batches of clusters and thus not causd by simple statistical fluctuations. In figure 2(b) for the simple cubic lattice, another sharp, but very different type of crossover is observed, where the small $|\ln \lambda|$ (or long time) region also gives a power-law but with a different exponent.

Let us first discuss the results shown in figure 2(a) for the square lattice in more detail. Here if only the flat region for larger $|\ln \lambda|$ is fitted to a power-law, the exponent comes out to be about 0.29 ± 0.01, 0.29 ± 0.01, and 0.19 ± 0.01 for $S = 2500$, 5000, and 10 000, respectively. We note that the last value would yield $d_w = 2.47 \pm 0.03$, which would be barely consistent with the value $d_w = 2.54 \pm 0.04$ obtained in [3] from the backbone statistics. This also means that these data would violate (1) very strongly, since $2d/d_w$ would be about 1.62 ± 0.02 using this value of d_w. Moreover, since the crossover point where the drop begins moves to smaller $|\ln \lambda|$ for larger cluster size S, we would expect this exponent to be the *correct* asymptotic exponent d_w for $S \to \infty$.

Our interpretation of this result is as follows: for sufficiently small time (thus length) scales ($t \ll \tau$) a typical random walk is trapped (by lack of loops) within order-one segment of the backbone (together with its dangling side branches) as proposed in [3], which results in a small d_w and the violation of (1). The crossover time τ should be such that

$$\tau^{1/d_w} \propto L \qquad (6)$$

where L is the length scale of the cluster and d_w is the walk dimension in this regime. Translated to the corresponding value in $|\ln \lambda|$, the crossover point should scale as

$$|\ln \lambda_1| \propto \tau^{-1} \propto L^{-d_w}. \tag{7}$$

However, as the walk goes beyond this time scale, it is forced by the finite size effect to sample more and more segments of the backbone. Thus over long times, an effective d_w increases, leading to the downward tendency in figure 2(a) beyond the crossover. Of course, for even longer times, the size of the visited region becomes limited by the total size of the cluster and the walk visits every site of the cluster; however, such a long-time regime is not reached in our figure.

Numerically, the crossover values $|\ln \lambda_1|$ for $\pi(\lambda)$ are approximately 5×10^{-4}, 2×10^{-4}, and 1×10^{-4} for $S=2500$, 5000, and 10 000, respectively. In comparison, L^{-d_w} with $L=\sqrt{S}/2$ and the estimate $d_w=2.47$ gives 3.5×10^{-4}, 1.5×10^{-4}, and 6.4×10^{-5}, in fair agreement with the discussion above. The discrepancy of about a factor of 1.5 in the absolute numbers should be due to the fact that our method is equivalent to summing over all random walks starting from all sites of the cluster and not just from the seed site at the centre.

Having identified d_w, we can return to the crossovers observed in figure 1(a) for $n(\lambda)$ in two dimensions. These crossovers occur roughly around $|\ln \lambda|=1\times10^{-4}$, 5×10^{-5}, and 2×10^{-5} for $S=2500$, 5000, and 10 000, respectively. The ratios of these numbers are again in fair agreement with those from (7).

The situation for $\pi(\lambda)$ in three dimensions is much less clear. In figure 2(b), $\pi(\lambda)$ is plotted for $S=2500$ and 5000 for the simple cubic lattice. Similarly to the square lattice, there is a flat region for larger $|\ln \lambda|$ which can be fitted to a power-law and what seems to be a crossover to another power-law at smaller $|\ln \lambda|$. The first region gives slopes of 0.49 ± 0.01 and 0.51 ± 0.01 for $S=2500$ and 5000, respectively. This would translate to d_w of about 3.92 ± 0.08 and 4.08 ± 0.09, respectively, and thus, with $d_f=d=3$, we would have $2d_f/d_w$ of around 1.5 which would yield a much smaller deviation from (1) than in two dimensions.

Beyond the crossover in the smaller $|\ln \lambda|$ region, we have another power-law with an exponent of about 0.32 and 0.29 for $S=2500$ and 5000, respectively. Since these exponents would lead to smaller values of d_w in this region, we would have an even stronger violation of the scaling relation (1).

The crossover point is at about $|\ln \lambda_1|=1\times10^{-3}$ and 4×10^{-4} for $S=2500$ and 5000, respectively. The ratio of these two numbers is in good agreement with (7) when a value of $d_w=4.08$ is used. Thus, we tentatively conclude that the asymptotic behaviour for an infinite Eden tree in three dimensions may also violate the condition (b) and thus (1) (but less strongly than in two dimensions). However, we do not understand the regime for $t \gg \tau$. One speculation would be that, in the latter regime, the random walk does not come to satisfy (b) over the whole cluster but becomes trapped in some parts of it, e.g. in the surface of the Eden tree. The walk dimension of this trapping region would then determine the slope after the crossover. However, the surface of the Eden model is a particularly difficult problem from the numerical point of view, with a very slow and non-monotonic convergence to the asymptotic behaviour [15]. It is also possible that the cluster sizes considered here are not yet in the asymptotic regime and that for much larger clusters a different picture emerges. Unfortunately no suitable technique is available to calculate d_s and d_w accurately on very large clusters.

3. Conclusion and discussion

In this work we have calculated the eigenspectrum of the random walk hopping probability matrix W and from this computed the critical exponents d_s and d_w for the Eden tree in two and three dimensions. By more accurate calculations, we confirm the violation of the condition (b) in section 1 on two-dimensional Eden trees, in agreement with [3], and moreover our data suggest that the same is true for three dimensions (although less strongly).

A natural question is why the detailed balance in this problem does not guarantee a *homogeneous* probability distribution within the expected travelling distance, since the latter may appear to be a consequence of *ergodicity*. However, ergodicity usually manifests itself as the evolution of any initial distribution of random walkers eventually into the stationary one. This is a statement about the ensemble of random walks of all starting points, and *not* about those of an individual starting point. Thus it is *not* necessary for an individual starting point to yield a homogeneous probability even in the $t \to \infty$ limit on an infinite Eden tree. Rather, it can so happen that the probability distribution for a given starting point is very anisotropic, sampling only order-one segment of the tree, but when we consider all possible starting points, the entire cluster is sampled, if one waits long enough.

It is also interesting to consider whether a similar situation arises in tree structures in general. A typical tree-like structure encountered in statistical physics is the diffusion limited aggregate (DLA) [1]. The available numerical results [16] from direct simulation of random walk displacements indicate that $d_w = 2.56 \pm 0.10$ in $d=2$ and 3.33 ± 0.25 in $d=3$, while the simulation of the return-to-origin probability in the same reference gives $d_s = 1.20 \pm 0.1$ in $d=2$ and 1.30 ± 0.1 in $d=3$. Using the known estimates [5] of $d_f = 1.68 \pm 0.05$ in $d=2$ and 2.5 ± 0.06 in $d=3$ together with the estimates for d_w from [16], we obtain the value of $2d_f/d_w$ to be about 1.31 ± 0.1 in $d=2$ and 1.50 ± 0.1 in $d=3$. While these values are not inconsistent with the direct estimates of d_s from [16], they are sufficiently different (particularly in $d=3$) to warrant further investigation.

Indeed, it may be possible that more or less all tree-like structures force an anisotropic probability distribution. If this were the case, it would be of great interest to develop a systematic theory of this anisotropy, perhaps with different coherence length exponents characterizing the radial and tangential directions in a manner somewhat similar to directed percolation [9]. This is, however, clearly out of the scope of this work and must await further research.

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